Math 245C Lecture 10 Notes

Daniel Raban

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1 Translation and Convolution

1.1 Translations of functions

Definition 1.1. Given $f : \mathbb{R}^n \to \mathbb{R}$ and $y \in \mathbb{R}^n$, define the **translation** $\tau_y f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$(\tau_y f)(x) = f(x - y).$$

Remark 1.1. If $1 \leq p \leq \infty$ and $y \in \mathbb{R}$, $\tau_y : L^p \to L^p$ is an isometry.

Remark 1.2. If $f \in \mathbb{R}^n \to \mathbb{R}$, then f is uniformly continuous if and only if the limit $\lim_{y \to 0} \|\tau_y f - f\|_u = 0$. Indeed,

$$\sup_{\|y\| \leq \delta} \|\tau_y - f\|_u = \sup_{\|z-y\| \leq \delta} |f(y) - f(z)|.$$

Remark 1.3. If $f : \mathbb{R}^n \to \mathbb{R}$ is supported by the ball $B_R(0)$, then

$$\|\tau_y f - f\|_p^p \leq |B_{R+1}(0)|^{1/p} \|\tau_y - f\|_\infty$$

whenever $\|y\| \leq 1$. Indeed,

$$\int_{\mathbb{R}^n} |f(x) - f(x-y)|^p \, dx = \int_{B_{R+1}(0)} |f(x) - f(x-y)|^p \, dx.$$

Let $C_c(\mathbb{R}^n)$ be the set of continuous functions $\mathbb{R}^n \to \mathbb{R}$ with compact support.

Lemma 1.1. *If $g \in C_c(\mathbb{R}^n)$, then g is uniformly continuous.*

Proof. Let $B_{R-1}(0)$ with $R > 1$ be a ball containing the support of $g \in C_c(\mathbb{R}^n)$. Then g is uniformly continuous on $\overline{B_R(0)}$. Set

$$\delta(r) = \sup_{\substack{\|x-y\| < r \\ \|x\|, \|y\| \leq R+1}} |g(x) - g(y)|.$$

We have

$$|g(x)-g(y)| \leq \begin{cases} 0 & \|y\| \geq R, \|x\| \geq R+1 \\ |g(x)-g(x_0)| & \|y\| < R, \|x\| \geq R+1 \\ 0 & \|x\| \geq R, \|y\| \geq R+1 \\ |g(x)-g(z_0)| & \|x\| \leq R, \|y\| \geq R+1. \end{cases}$$

Consequently,

$$\sup_{\|x-y\| \leq r} |g(x)-g(y)| \leq \delta(r).$$

So g is uniformly continuous. $\square$

Proposition 1.1. *If $1 \leq p < \infty$, then τ_y converges pointwise to the identity map in L^p:*

$$\lim_{y \to 0} \|\tau_y f - f\|_p = 0.$$

Proof. Let $f \in L^p$. For any $g \in C_c(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, we have

$$\begin{aligned} \|\tau_y f - f\|_p &\leq \|\tau_y f - \tau_y g\|_p + \|\tau_y g - g\|_p + \|g - f\|_p \\ &= 2\|f-g\| + \|\tau_y g - f\|_p \\ &\leq 2\|f-g\| + |B_R|^{1/p} \|\tau_y g - f\|_u, \end{aligned}$$

where $B_{R-1}(0)$ is a ball containing the support of g. Since g is uniformly continuous, we conclude

$$\limsup_{y \to 0} \|\tau_y f - f\|_p \leq 2\|f-g\|_p.$$

Since $C_c(\mathbb{R}^n)$ is dense in L^p,

$$\limsup_{y \to 0} \|\tau_y f - f\|_p = 0. \qquad \square$$

1.2 Convolution

Definition 1.2. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be measurable, and let $x \in \mathbb{R}^n$ be such that $y \mapsto \tau_y f g$ is integrable. Then we define the **convolution** of f and g as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy = \int \tau_y f(x) g(y)\,dy.$$

Definition 1.3. The **n-torus** is $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$.

If $x \in \mathbb{R}^n$, the equivalence class of x in $\mathbb{T}^n$ is $x + \mathbb{Z}^n = \hat{x}$. The metric on $\mathbb{T}^n$ is

$$\|\hat{x} - \hat{y}\|_{\mathbb{T}^n} = \inf_{z \in \mathbb{Z}^n} |x - y - z|.$$

There is a bijection between $\mathbb{T}^n$ and $Q_n = [-1/2, 1/2)^n$. Consequently, there is a bijection between $\mathbb{T}^n$ and $\tilde{Q}_n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| = 1 \; \forall i\}$. Since $\tilde{Q}_n$ is compact, we conclude that $\mathbb{T}^n$ is a compact set.

Proposition 1.2. *If $x \in \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}$ are masurable, and $y \mapsto \tau_y f(x) g(y)$ is integrable, then*

$$(f * g)(x) = (g * f)(x).$$

Proof. Use the change of variables $z = x - y$:

$$\begin{aligned}
(f * g)(x) &= \int_{\mathbb{R}^n} f(x-y) g(y)\, dy \\
&= \int_{\mathbb{R}^n} f(z) g(x-z)\, dz \\
&= (g * f)(x).
\end{aligned}$$

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